## Pythagorean theorem:-

QN→ If $x$ & $y$ are any two orthogonal vectors in an inner product space $E$ (or in a Hilbert space) then, $\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2$.

**Proof:-** Let $x \perp y$. Then $(x/y) = 0$. Also $(y/x) = 0$.

Now, $\|x+y\|^2 = (x+y/x+y) = (x/x) + (y/x) + (x/y) + (y/y)$

$$= \|x\|^2 + 0 + 0 + \|y\|^2$$

$$= \|x\|^2 + \|y\|^2.$$

$\|x-y\|^2 = (x-y/x-y) = (x/x) - (y/x) - (x/y) + (y/y)$

$$= \|x\|^2 - 0 - 0 + \|y\|^2$$

$$= \|x\|^2 + \|y\|^2.$$

QN→ For any non-empty subset $S$ of a Hilbert space $H$, the orthogonal complement of $S$ i.e. $S^\perp$ is a closed linear subspace of $H$ and hence $S^\perp$ is a Hilbert space.

**Proof:-** Let $x, y \in S^\perp$ and $\alpha, \beta$ be any scalars.
Let $z \in S$ be arbitrary, then

$$(\alpha x + \beta y / z) = \alpha (x/z) + \beta (y/z)$$

$$= \alpha \cdot 0 + \beta \cdot 0 = 0$$

Hence, $\alpha x + \beta y$ is orthogonal to $S$.
Therefore, $\alpha x + \beta y \in S^\perp$. Hence $S^\perp$ is a linear subspace of $H$. Let $u$ be any accumulation

Point of $S^\perp$ in $H$. We shall show that $u \in S^\perp$. By the property of accumulation point in a metric space, there exists a sequence $(x_n)$ of points of $S^\perp$ such that $x_n \to u$.

Let $z \in S$ be arbitrary. Then by the continuity of the inner product function, $(x_n/z) \to (u/z)$. But since $x_n \in S^\perp$, $(x_n/z) = 0$ for every $z \in S$. Hence $(u/z) = \lim(x_n/z) = 0$ for every $z \in S$.

Hence, $u \in S^\perp$. Therefore, $S^\perp$ is a closed linear subspace of $H$. Since $H$ is complete so $S^\perp$ is also complete. Hence $S^\perp$ is a Hilbert space.

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(Lemma of F. Riesz on closed convex set in a Hilbert space)

Q No → Prove that a closed convex subset of a Hilbert space contains a unique vector of smallest norm.

or, Q No → Let $A$ be a closed convex set in a Hilbert space $H$. Then there exists a unique $x_0 \in A$ such that $\|x_0\| \leq \|y\|$ for every $y \in A$.

Proof:- Let $B = \text{glb}\{\|x\| : x \in A\}$. Then for each +ve integer $n$, there exists $x_n \in A$ such that

$$B \leq \|x_n\| < B + \frac{1}{n}.$$

Hence the sequence $(x_n)$ in $A$ is such that $\|x_m\| \to \beta$. Since A is Convex,

$$\frac{x_m + x_n}{2} \in A \text{, and hence, } \left\|\frac{x_m + x_n}{2}\right\| \geq \beta$$

By the Parallelogram law, we obtain,

$$\|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2$$

$$\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4\beta^2.$$

$$\to 2\beta^2 + 2\beta^2 - 4\beta^2 = 0 \text{ as } m, n \to \infty.$$

hence, $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Therefore, $(x_n)$ is a Cauchy sequence in A. Since A is a Closed set in the Complete metric space H, A is Complete. Hence there exists $x_0 \in A$ such that $x_n \to x_0$. Since norm is Continuous function $\|x_n\| \to \|x_0\|$. But $\|x_n\| \to \beta$.

Hence on account of uniqueness of limit, $\|x_0\| = \beta$. To show that $x_0$ is the unique such vector, let $x_1$ be a vector in A different from $x_0$ such that,

$\|x_1\| = \beta$. Then $\frac{x_0 + x_1}{2} \in A$ and hence

$$\left\|\frac{x_0 + x_1}{2}\right\| \geq \beta \quad \text{--- (1)}$$

But by Parallelogram law,

$$\left\|\frac{x_0 + x_1}{2}\right\|^2 = \frac{1}{2}\|x_0\|^2 + \frac{1}{2}\|x_1\|^2 - \left\|\frac{x_0 - x_1}{2}\right\|^2$$

$$< \frac{1}{2}\|x_0\|^2 + \frac{1}{2}\|x_1\|^2 \text{ (Since, } \|x_0 - x_1\| > 0$$

$$= \frac{1}{2}\beta^2 + \frac{1}{2}\beta^2 = \frac{\beta^2+\beta^2}{2} = \frac{\cancel{2}\beta^2}{\cancel{2}}$$

$$= \beta^2.$$

Thus $\left\| \frac{x_0 + x_1}{2} \right\| < \beta$ —— (2)

(1) & (2) Contradict each other. Hence $x_0$ is the unique vector in A with smallest norm.

(Def<sup></sup>) Let E be an inner Product Space and M be a Subset of E. For a given $x$ in E, an element $y_0$ of M is Said to be a best approximation to $x$ from M if for all $z \in M$,

$$\|x - y_0\| \leq \|x - z\|$$

Such $y_0$ is also Said to be an optimal Solution of the Problem.

Minimize $\|x - z\|$, $z \in M$, $x - y_0$ is called an optimal error.

QNo> Let M be a closed linear Subspace of a Hilbert Space H and let $x$ be a vector in H Such that $x$ is not in M and let d be the distance from $x$ to M i.e. $d = \inf\{\|x - z\| : z \in M\}$ Then there exists a unique vector $y_0$ in M Such that $\|x - y_0\| = d$. i.e. there is a unique best approximation $y_0$ to $x$ from M. Moreover, this $y_0$ is the unique element of M for which $x - y_0$ is orthogonal to M.

Proof:- Let $x \in H$ and $d = \inf\{\|x - z\| : z \in M\}$. Then for each +ve integer $n$ there exists $z_n \in M$ such that

$$d \leq \|x - z_n\| < d + \frac{1}{n}.$$

Hence the sequence $(z_n)$ in $M$ is such that

$$\|x - z_n\| \to d.$$

Since $M$ is a linear subspace of $H$, $M$ is a convex set. Hence $\frac{(z_n + z_m)}{2} \in M$ for all $n$ & $m$. Therefore,

$$\left\|x - \frac{(z_n + z_m)}{2}\right\| \geq d.$$

Now by the Parallelogram law applied to $z_n \to x$ and $z_m \to x$, we have

$$\|(z_n + z_m) - 2x\|^2 + \|z_n - z_m\|^2$$
$$= 2\|z_n - x\|^2 + 2\|z_m - x\|^2$$

Hence, $\|z_n - z_m\|^2 \leq 2\|z_n - x\|^2 + 2\|z_m - x\|^2 - 4d^2$

$$\to 2d^2 + 2d^2 - 4d^2 = 0 \text{ as } n, m \to \infty.$$

Hence, $z_n$ is a Cauchy sequence in $M$. Since $M$ is a closed set in the complete metric space $H$ so $M$ is complete. Hence there exists $y_0$ in $M$ such that $z_n \to y_0$. Hence, $x - z_n \to x - y_0$. Since norm is a continuous function,

$$\|x - z_n\| \to \|x - y_0\|$$

Thus, $\|x - y_0\| = \text{Lt.} \|x - z_n\| = d.$

Thus there exists $y_0 \in M$ such that

$$\|x - y_0\| = d.$$

This shows that $y_0$ is a best approximation to $x$

from M. In order to show that $y_0$ is unique let $y_1$ be a vector in M such that,

$$\|x - y_1\| = d.$$

Then $\frac{(y_0 + y_1)}{2} \in M$ and hence,

$$\left\| x - \frac{(y_0 + y_1)}{2} \right\| \geq d \quad \text{--- (1)}$$

But by Parallelogram law applied to $y_0 - x$ & $y_1 - x$ we have

$$\|(y_0 + y_1) - 2x\|^2 + \|y_0 - y_1\|^2$$
$$= 2\|y_0 - x\|^2 + 2\|y_1 - x\|^2$$

Hence, $\|y_0 - y_1\|^2 \leq 2\|y_0 - x\|^2 + 2\|y_1 - x\|^2 - 4d^2$
$$= 2d^2 + 2d^2 - 4d^2 = 0.$$

Thus $\|y_0 - y_1\| = 0$. hence $y_0 = y_1$.

Hence there exists unique $y_0 \in M$ such that

$$\|x - y_0\| = d.$$

Thus there is a unique best approximation $y_0$ to $x$ from M.

To show that $(x - y_0) \in M^\perp$ Consider $z \in M$ with $\|z\| = 1$.

Then, $\omega = y_0 + (x - y_0, z) z \in M$.

and, we have

$$\|x - y_0\|^2 \leq \|x - \omega\|^2$$
$$= (x - \omega, x - \omega)$$
$$= ((x - y_0) - (x - y_0, z)z, (x - y_0) - (x - y_0, z)z)$$
$$= (x - y_0, x - y_0) - |(x - y_0, z)|^2$$
$$- |(x - y_0, z)|^2 + |(x - y_0, z)|^2 \cdot \|z\|^2.$$

$$= \|x - y_0\|^2 - |(x - y_0, z)|^2 \quad [\because \|z\| = 1].$$

This shows that,

$$(x - y_0, z) = 0 \text{ i.e. } (x - x_0) \perp z.$$

Hence, $(x - y_0)$ is orthogonal to $M$.

Conversely, let $y_0 \in M$ and $(x - y_0) \perp M$. Then for any $z \in M$, we have $y_0 - z \in M$ so that $(x - y_0) \perp (y_0 - z)$.

Hence, by Pythagoras theorem,

$$\|x - z\|^2 = \|(x - y_0) + (y_0 - z)\|^2$$

$$= \|x - y_0\|^2 + \|y_0 - z\|^2$$

Thus, $\|x - y_0\| < \|x - z\|$ if $y_0 \neq z$.

This also shows that $y_0$ is the unique best approximation to $x$ from $M$.

This completes the proof.